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NILPOTENT SUBGROUPS OF THE GROUP OF SELF-HOMOTOPY EQUIVALENCES

Dedicated to Professor Akio Hattori on his sixtieth birthday

BY

KEN-ICHI MARUYAMA[a],† AND MAMORU MIMURA[b]

[a]*Department of Mathematics, Faculty of Education, Chiba University, Yayoicho Chiba, Japan; and* [b]*Mathematical Sciences Research Institute, Berkeley, California, USA and Department of Mathematics, Faculty of Science, Okayama University, Okayama, Japan*

ABSTRACT

In this paper, we study subgroups of self-homotopy equivalences associated to generalized homology theories. We generalize Dror–Zabrodsky's nilpotency theorem on the group of self-homotopy equivalences.

§1. Introduction

Let $\mathcal{E}(X)$ be the group of homotopy classes of self-homotopy equivalences of a space X (preserving the base point). When E is an arbitrary homology theory, we denote by $\mathcal{E}_E(X)$ the kernel of the following natural map:

$$\mathcal{E}(X) \to \prod_{i=-\infty}^{\infty} \mathrm{Aut}(E_i(X)).$$

The following theorem is shown by Dror–Zabrodsky in [4].

THEOREM (Theorem D of [4]). *Let X be a nilpotent finite dimensional space and $G \subset \mathcal{E}(X)$ a subgroup. If G acts nilpotently on $H_i(X, Z)$ for all $i \leq \dim X$, then G is nilpotent.*

Thus $\mathcal{E}_H(X)$ is nilpotent for a space X as above, where H is the ordinary homology theory.

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Our object is to generalize the above theorem for a generalized homology theory. The following is our main theorem:

THEOREM 1.1. *Let E be a connective, reduced homology theory, represented by a CW ring spectrum such that* $\pi_0(E)$ *is isomorphic to* Z_P, *where* Z_P *is the integer localized at some set of primes, P.*

Then a subgroup G of $\mathcal{E}(X_P)$ *acting on* $E_*(X_P)$ *nilpotently is a nilpotent group if X is a connected, finite dimensional nilpotent space.*

COROLLARY 1.2. *If* $E = MU$ *or* MSp, *then* $\mathcal{E}_E(X)$ *is a nilpotent group for a space X satisfying the conditions in the above theorem.*

PROOF. As is well known (see [2]), MU and MSp are multiplicative. It is also known that $\pi_0 MU \simeq Z$ and $\pi_0 MSp \simeq Z$, and hence clearly the conditions of Theorem 1.1 are satisfied. Q.E.D.

COROLLARY 1.3. *Let p be a prime. If* $E = BP$ *or one of the theories in the above localized at p, then* $\mathcal{E}_E(X_{(p)})$ *is a nilpotent group for a space X satisfying the above conditions.*

PROOF. It is known (see [2]) that BP is a ring spectrum and $\pi_* BP$ is a polynomial ring over $Z_{(p)}$. Therefore we can apply Theorem 1.1. Similarly for the other cases. Q.E.D.

REMARK. We do not know whether or not the condition "ring spectrum" in the theorem can be dropped.

The paper is organized as follows. In §2 we recall the notion of nilpotent actions. In §3 we prove the main theorem, Theorem 1.1. In §4 we give some counter examples.

§2. Nilpotent actions

In this section we recall a notion of a nilpotent action following [5].

Let A be an abelian group and $\omega : Q \to \mathrm{Aut}(A)$ an action of a group Q on A. We define inductively

$$\Gamma^1_\omega(A) = A, \quad \Gamma^{i+1}_\omega(A) = \langle \omega(x)\cdot a - a \mid x \in Q, \quad a \in \Gamma^i_\omega(A) \rangle \quad \text{for } i > 1.$$

The action of Q is said to be nilpotent if there is some j such that $\Gamma^j_\omega(A) = \{0\}$. The maximal integer c such that $\Gamma^c_\omega(A) \neq 0$ is called the nilpotency of the action, denoted by $\mathrm{nil}(\omega) = c$. We will mention two properties without proof.

PROPOSITION 2.1 (Proposition 4.3 of [5]). *Let $A' \to A \to A''$ be an exact sequence of Q-modules. If the Q-actions on A' and A'' are nilpotent, so is the Q-action on A.*

PROPOSITION 2.2 (Proposition 4.15 of [5]). *Let $\omega : Q \to \mathrm{Aut}(A)$ be a nilpotent action. If $F: Ab \to Ab$ is a half exact functor of the category of abelian groups, then the induced action $F\omega : Q \to \mathrm{Aut}(FA)$ is also a nilpotent action.*

§3. Proof of the main theorem

Our argument depends entirely upon the results in [4]. In fact, we will show that the subgroup $G \subset \mathcal{E}(X_P)$ in Theorem 1.1 acts nilpotently on the ordinary homology.

Let Y be a CW complex and denote by $\mathcal{Y}$ its suspension spectrum. According to Adams (pp. 316–317 of [2]), for any ring spectrum E, there exists an E_*-Adams spectral sequence $\{E_r^{*,*}\}$ such that

$$E_1^{s,t} \simeq \pi_{t-s}(E \wedge \bar{E}^s \wedge \mathcal{Y}),$$

where $\bar{E}$ is a spectrum defined by a cofibration $\bar{E} \to S \xrightarrow{\eta} E$ with the unit $\eta : S \to E$ and $\bar{E}^s = \bar{E} \wedge \cdots \wedge \bar{E}$ with s factors.

Furthermore, by Bousfield [3], the spectral sequence converges to $\pi_*^S(Y) \otimes Z_P = \pi_*^S(Y)_P$ as follows, if the last condition, $\pi_0(E) \simeq Z_P$, is satisfied: There is a filtration

$$\pi_n^S(Y)_P = F^{0,n} \supset F^{1,n+1} \supset \cdots$$

such that we have the following natural homomorphisms:

$$\pi_n^S(Y)_P \to \varprojlim \pi_n^S(Y)_P / F^{*,*+n}. \tag{3.1}$$

$$F^{s,t}/F^{s+1,t+1} \to E_\infty^{s,t}. \tag{3.2}$$

The following is due to Bousfield:

THEOREM 3.3 (Theorem 6.5 of [3]). *Let E be a connective ring spectrum such that $\pi_0 E \simeq Z_P$. Then*

(1) *the homomorphisms in the above are isomorphic, that is, the spectral sequence $\{E_r^{*,*}\}$ converges completely;*

(2) *for every s and t there is a sufficiently large N such that $E_\infty^{s+N,t+N} = 0$, that is, the spectral sequence $\{E_r^{*,*}\}$ is strongly Mittag-Leffler in the sense of Bousfield.*

Now consider the case where $Y = X_P$. There is the Atiyah–Hirzebruch spectral sequence $\{E^r_{*,*}\}$ such that

$$E^2_{*,*} = H_*(\bar{E}^s, (E \wedge \mathfrak{X}_P)_*).$$

Since $\bar{E}^s$ is connective, this spectral sequence converges (see [2]), that is, there exists a filtration

$$(E \wedge \mathfrak{X}_P)_{p+q}(\bar{E}^s) \supset \cdots \supset J_{p,q} \supset \cdots \supset J_{0,p+q} \supset J_{-1,p+q+1} = \{0\},$$

such that

$$(E \wedge \mathfrak{X}_P)_{p+q}(\bar{E}^s) = \bigcup J_{i,j} \quad (i + j = p + q) \quad \text{and} \quad E^\infty_{p,q} = J_{p,q}/J_{p-1,q+1}.$$

Here recall that $(E \wedge \mathfrak{X}_P)_*(\bar{E}^s) \simeq E_1^{s,*}$.

Let G be an arbitrary subgroup of $\mathcal{E}(X_P)$ acting on $E_*(X_P)$ nilpotently as in the assumption of the theorem. Then by the naturality of the spectral sequence G acts on $E_r^{s,t}$. First G acts on $H_*(\bar{E}^s, (E \wedge \mathfrak{X}_P)_*)$ nilpotently by Proposition 2.2, and hence, using the filtration $\{J_{p,q}\}$, we inductively see that G acts on $(E \wedge \mathfrak{X}_P)_*(\bar{E}^s) \simeq E_1^{s,*}$ nilpotently. Since $E_{r+1}^{*,*}$ is the homology of $E_r^{*,*}$, the action of G on it is also nilpotent by Proposition 2.2. Again inductively one can see that G acts on $E_\infty^{*,*}$ nilpotently.

By Theorem 3.3 we have the following short exact sequence:

$$0 \to F^{s+1,t+1} \to F^{s,t} \to E_\infty^{s,t} \to 0.$$

Also by the naturality the above exact sequence is seen to be an exact sequence of G-modules. As stated above, G acts nilpotently on the E_∞-term and so, if G acts nilpotently on $F^{s+1,t+1}$, then one can see by Proposition 2.1 that G acts nilpotently on $F^{s,t}$ in the middle. Since $F^{s+N,t+N}$ is trivial for a sufficiently large N, by Theorem 3.3, one can easily see that G acts nilpotently on $F^{0,t-s} = \pi^S_{t-s}(X_P)$ by induction; that is, G acts nilpotently on $\pi^S_*(X_P)$.

LEMMA 3.4. *Let Y be a nilpotent space. If a subgroup $A \subset \mathcal{E}(Y)$ acts nilpotently on the homotopy group $\pi_i(Y)$ for $i \leq \dim Y$, then A acts nilpotently also on the ordinary homology group of Y.*

PROOF. As in the argument on page 189 of [4], the lemma can be shown inductively by using the Serre spectral sequence

$$E_2^{s,t} = H_s(Y_{i-1}, H_t(K(\pi_i(Y), i)) \Rightarrow H_*(Y_i)$$

which is obtained from the Postnikov decomposition $\{Y_i\}$ of Y. Q.E.D.

Consider the homomorphism

$$\Sigma^n_\#: \mathcal{E}(X_P) \to \mathcal{E}(\Sigma^n X_P)$$

induced from the suspension. We have

$$\pi_i^S(X_P) = \pi_{N+i}(\Sigma^N X_P), \qquad i = 1, \ldots, \dim X$$

for a sufficiently large integer N. The subgroup $\Sigma^N_\#(G) \subset \mathcal{E}(\Sigma^N X_P)$ acts nilpotently on the i-th homotopy group $\Sigma^N X_P$ for $i \leq \dim \Sigma^N X$ and hence by Lemma 3.4 $\Sigma^N_\#(G)$ acts nilpotently also on $H_*(\Sigma^N X_P)$.

Now we denote the actions

$$G \to \Pi\,\mathrm{Aut}(H_i(X_P)) \quad \text{and} \quad \Sigma^N_\#(G) \to \Pi\,\mathrm{Aut}(H_{N+i}(\Sigma^N X_P))$$

by ω and ω', respectively. Then the following diagram is commutative:

$$\begin{array}{ccc} G & \xrightarrow{\omega} & \Pi\,\mathrm{Aut}(\tilde{H}_i(X_P)) \\ \Sigma^N_\# \downarrow & & \downarrow = \\ \Sigma^N_\#(G) & \xrightarrow{\omega'} & \Pi\,\mathrm{Aut}(\tilde{H}_{i+N}(\Sigma^N X_P)) \end{array}$$

from which follows $\Gamma^i_\omega \tilde{H}_*(X_P) \simeq \Gamma^i_{\omega'} \tilde{H}_{*+N}(\Sigma^N X_P)$. Since ω' is nilpotent, as was already seen, we have $\Gamma^i_\omega \tilde{H}_*(X_P) = 0$ for some i. Thus the subgroup $G \subset \mathcal{E}(X_P)$ acts nilpotently on the homology. By the Dror–Zabrodsky theorem (Theorem D of [4]) G is a nilpotent group. This completes the proof of Theorem 1.1.

Quite similarly, one can prove the following:

THEOREM 3.5. *Let E be a connective CW ring spectrum such that $\pi_0 E \simeq Z/p$ with p a prime. Let X be a connected, finite dimensional nilpotent space such that $H_i(X)$ is a finite p-torsion group for any $i \geq 1$. Then, if a subgroup $G \subset \mathcal{E}(X)$ acts nilpotently on $E_*(X)$, G is a nilpotent group.*

PROOF. The argument is parallel to the above one, using the E_*-Adams spectral sequence. In this case the spectral sequence converges to $\varprojlim \pi_i^S(X)/p^t\pi_i^S(X)$ (cf. Theorem 6.6 of [3]).

Here we have

$$\varprojlim \pi_i^S(X)/p^t\pi_i^S(X) \simeq \pi_i^S(X),$$

since $\pi_i^S(X)$ is a finite p-torsion group. Also, we have

$$F^{s+N,t+N} = 0 \qquad \text{for a sufficiently large } N$$

by the same condition. Now the proof is completed by repeating the above argument, since G acts nilpotently on $\pi_i^S(X)$. Q.E.D.

Let $I_n \subset \pi_* BP$ be a BP^*BP-invariant prime ideal. Using the Bass–Sullivan techniques one may construct a BP-module spectrum $P(n)$ with the property that $\pi_* P(n) = BP/I_n$. We also consider $k(n)$, the connective Morava extraordinary K-theory. (See [6], [7] for details of them.)

COROLLARY 3.6. *Let $E = HZ/p$, MO $(p = 2)$, $P(n)$, $k(n)$. If X satisfies the assumption of Theorem* 3.5, *then $\mathcal{E}_E(X)$ is a nilpotent group.*

PROOF. (1) The case $E = HZ/P$ is clear.

(2) The case $E = MO$ (for $p = 2$) is also clear, since the ring spectrum MO satisfies $\pi_0 MO \simeq Z/2$ (see [2]).

(3) The case $E = P(n)$; it is known that $P(n)$ is a multiplicative theory and $\pi_* P(n) = BP/I_n$, where $I_n = (p, v_1, \ldots, v_n)$.

(4) The case $E = k(n)$; it is known that $k(n)$ is a multiplicative theory and $\pi_* k(n) = Z/p[v_n]$. Q.E.D.

§4. Some counter examples

It is not always true that $\mathcal{E}_E(X)$ is a nilpotent group for any homology theory E. For example, there is an example for the K-theory as follows: Let p be an odd prime and $\alpha_1 \in {}^p\pi_{2p-3}^S(S^0) \simeq Z/p$ a generator. Then there is a map α for which the following diagram is homotopy commutative:

$$\begin{array}{ccc} S^{n+2p-2} \bigcup_p e^{n+2p-1} & \xrightarrow{\alpha} & S^n \bigcup_p e^{n+1} \\ \uparrow & & \downarrow \\ S^{n+2p-2} & \xrightarrow{\alpha_1} & S^{n+1} \end{array}$$

for a sufficiently large n. Usually the mapping cone of α is denoted by $V(1)$. It is known (see [1]) that α induces a K_*-isomorphism and hence $V(1)$ is a K_*-acyclic space. Put

$$mX = V(1) \vee \cdots \vee V(1) \qquad \text{(wedge of } m\text{-copies of } V(1)).$$

Since $\tilde{K}_*(mX) = 0$, we have $\mathcal{E}_{\tilde{K}}(mX) = \mathcal{E}(mX)$. On the other hand, $\mathcal{E}(mX)$ contains a subgroup isomorphic to the symmetric group S_m. As is well known, S_m is not nilpotent for $m > 2$. Therefore $\mathcal{E}_{\tilde{K}}(mX) = \mathcal{E}(mX)$ is not nilpotent either.

In general, let Y be a space and consider $\mathcal{E}(mX \vee Y)$. Then $\mathcal{E}_{\tilde{K}}(mX \vee Y)$ contains a subgroup $\mathcal{E}_{\tilde{K}}(mX) \simeq \mathcal{E}(mX)$ and hence it is not a nilpotent group.

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